Math 155, ***Lecture Notes- Bonds*** **Name____________**

Section 9.5 Alternating Series

Up until now, we have been studying series with positive terms. In this section we will study series that have both positive and negative terms. In particular, we will start by consider series that have terms that alternate in and we will call these **alternating series.** In order to prove convergence, or divergence for these series, we will still need to use our knowledge of the convergence and divergence characteristics of geometric series, harmonic series, *p*-series, and telescoping series.

Alternating series can be written in two ways, either the even terms are negative, or the odd terms are negative. Below, we have a convergence test for alternating series:

THEOREM 9.14 Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ **2.** $a_{n+1} \le a_n$, for all n

Ex. 1: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, with $\frac{1}{n} = a_n$.

Let's use the Alternating Series Test to show Convergence for the series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, We need to show $a_n > 0$. We know $1 > 0$ and $n > 0$ for all $n \ge 1$, we can see that $\frac{1}{n} > 0$ since a ratio of positive numbers is positive, (RoPNiP)

This means that $a_n > 0$ for all $n \ge 1$.

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$

$= 0.$

More Ex. 1:

Now, we need to show $a_{n+1} \leq a_n$ for all $n \geq 1$.

We know $n \leq n+1$ for all $n \geq 1$.

we can see $\frac{1}{n(n+1)} \cdot \left(\frac{n}{1}\right) \leq \frac{1}{n(n+1)} \cdot \left(\frac{n+1}{1}\right)$

$$\frac{1}{n+1} \leq \frac{1}{n}$$

and $a_{n+1} \leq a_n$.

Therefore, according to the Alternating Series Test, the series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, converges. $\square$

Ex. 2: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$

Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, with $a_n = \frac{n}{n^2+1}$.

We're going to use the Alternating Series Test to show convergence for the series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$.

We need to show $a_n > 0$. We know $n > 0$ and $n^2+1 > 0$ for all $n \geq 1$. We can see that $\frac{n}{n^2+1} > 0$ since a ratio of positive numbers is positive. (RoPNip)

This means that $a_n > 0$ for all $n \geq 1$.

Consider
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1}$$
$$= \lim_{n\to\infty} \left[\frac{\frac{n}{1}}{\frac{n^2+1}{1}}\right] \cdot \left[\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right]$$
$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}}$$
$$= \frac{0}{1+0}$$
$$= 0.$$

Now, we need to show $a_{n+1} \leq a_n$ for all $n \geq 1$.

We know
$$n^3+n^2+n+1 \leq n^3+2n^2+n+n$$
$$n^3+n^2+n+1 \leq n^3+2n^2+2n$$
$$(n^2+1)(n+1) \leq n(n^2+2n+2)$$
$$\frac{1}{(n^2+1)(n^2+2n+2)} \cdot \left[\frac{(n^2+1)(n+1)}{1}\right] \leq \frac{1}{(n^2+1)(n^2+2n+2)} \left[\frac{n(n^2+2n+2)}{1}\right]$$
$$\frac{n+1}{n^2+2n+2} \leq \frac{n}{n^2+1}$$
$$\frac{n+1}{(n^2+2n+1)+1} \leq \frac{n}{n^2+1}$$

More Ex. 2:

$$\frac{n+1}{(n+1)^2+1} \le \frac{n}{n^2+1}$$

and $a_{n+1} \le a_n$.

Therefore, according to the Alternating Series Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$ converges. $\square$

Ex. 3: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$

Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, with $\frac{n^2}{n^2+5} = a_n$.

If we try to use the Alternating Series Test to show convergence for the series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$, we will run into a problem.

Consider

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+5}$$

$$= \lim_{n\to\infty} \left[\frac{\frac{n^2}{1}}{\frac{n^2+5}{1}}\right] \cdot \left[\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{5}{n^2}}$$

$$= \frac{1}{1+0}$$

$$= 1$$

Since this limit is not zero, we can't use the A.S.T. to show convergence for the series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$. In fact, after looking at the graph of the sequence of partial sums, and after considering previous limit value, we should try to show that the series diverges. The A.S.T. cannot be used to show divergence. We use the nth-term Test for Divergence to show that the series diverges. Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5} = \sum_{n=1}^{\infty} a_n$, with $\frac{(-1)^{n+1} n^2}{n^2+5} = a_n$. Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$.

Due to oscilation, the $\lim_{n\to\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$ does not exist.

Since $\lim_{n\to\infty} a_n \neq 0$, the nth-Term Test for Divergence tells us that the series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+5}$, diverges.

$\square$

Ex. 4: Consider: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Earlier, we saw that this series was convergent.
Can we approximate its sum?

Consider $S_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}$

$S_{10} = \frac{1627}{2520}$

$S_{10} \approx 0.64563$

We can say $S_{10} \approx S$, where $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
But, what is the error in our approximation?

$S = S_N + R_N$ & $R_N = S - S_N$
where R_N is the remainder

THEOREM 9.15 Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}.$$

Proof: Let $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ be a convergent Alternating Series satisfying ① $\lim_{n\to\infty} a_n = 0$ and ② $a_{n+1} \le a_n$

Consider $R_N = S - S_N$, $a_n \geq 0$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^{N} (-1)^{n+1} a_n$$

$$= (-1)^{N+2} a_{N+1} + (-1)^{N+3} a_{N+2} + (-1)^{N+4} a_{N+3} + (-1)^{N+5} a_{N+4} + \cdots$$

$$= (-1)^{N+2} [a_{N+1} + (-1)^1 a_{N+2} + (-1)^2 a_{N+3} + (-1)^3 a_{N+4} + \cdots]$$

$$|R_N| = |(-1)^{N+2} \cdot [a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + \cdots]|$$

$$|R_N| = a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + \cdots$$

$$= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) + \cdots \leq a_{N+1}$$

So,

$$|R_N| \leq a_{N+1}.$$

Back to $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

with $S_{10} \approx S$,

and $S \approx 0.64563$

we'll have $R_{10} = \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \cdots$,

so

$$R_{10} = \frac{1}{11} - \left(\frac{1}{12} - \frac{1}{13}\right) - \left(\frac{1}{14} - \frac{1}{15}\right) - \cdots \leq \frac{1}{11}$$

this means the S_{10} is within $\frac{1}{11}$ of S.

or, $|S - S_{10}| \leq \frac{1}{11}$

$$-\frac{1}{11} \leq S - 0.64563 \leq \frac{1}{11}$$

$$0.64563 - \frac{1}{11} \leq S \leq \frac{1}{11} + 0.64563$$

$$\boxed{0.64563 \leq S \leq 0.73654}$$

Ex. 5: (a) Use *Theorem 9.15* to determine the number of terms required to approximate the sum of the convergent series with an error of less than 0.001.
(b) Use a graphing utility approximate the sum of the series with an error if less than 0.001.

Consider: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos(1)$

(a) The approximation error is the remainder. By Theorem 9.15 we have

$|R_N| \le a_{N+1}$, with $a_n = \frac{1}{(2n)!}$

So $|R_N| \le a_{N+1} = \frac{1}{[2(N+1)]!} < 0.001$

If we solve for N, we can find the number of terms we need to know S_N is within 0.001 of S.

Solve for N: $\frac{1}{(2N+2)!} < \frac{1}{1{,}000}$

$N=1$, $\frac{1}{(2+2)!} = \frac{1}{4!} = \frac{1}{1\cdot2\cdot3\cdot4} = \frac{1}{24}$

$N=2$, $\frac{1}{(4+2)!} = \frac{1}{6!} = \frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot6} = \frac{1}{720}$

$N=3$, $\frac{1}{(6+2)!} = \frac{1}{8!} = \frac{1}{1\cdot2\cdot3\cdot4\cdot5\cdot7\cdot8} = \frac{1}{40{,}320}$

So, $\frac{1}{[2(N+1)]!} < 0.001$ when $N \ge 3$.

Since we start with $n=0$ in $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$, we'll

(a) use 4 terms when $N=3$ for S_3.

cont'd

$$S_3 = \frac{(-1)^0}{[2(0)]!} + \frac{(-1)^1}{[2(1)]!} + \frac{(-1)^2}{[2(2)]!} + \frac{(-1)^3}{[2(3)]!}$$

$$S_3 = \frac{1}{0!} + \frac{(-1)}{2!} + \frac{1}{4!} + \frac{(-1)}{6!}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$$

$$S_3 = \frac{389}{720}$$

(b) $S_3 = 0.5402\overline{7}$, or $S_3 \approx 0.5403$

$\cos(1) \approx 0.5403023059$

THEOREM 9.16 Absolute Convergence

If the series $\Sigma|a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

Proof: If $\Sigma|a_n|$ converges, then $\Sigma\, 2\cdot|a_n|$ converges.
For all n , we can see that

$$0 \le a_n + |a_n| \le 2|a_n|.$$

So, by Theorem 9.12, the Direct Comparison Test,
$\Sigma(a_n + |a_n|)$ converges because
$\Sigma\, 2|a_n|$ converges.
By writing $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n|$
we can see that the series $\sum_{n=1}^{\infty} a_n$ converges,
since both $\sum_{n=1}^{\infty}(a_n + |a_n|)$ and $\sum_{n=1}^{\infty}|a_n|$ are
convergent series.

Definitions of Absolute and Conditional Convergence

1. $\Sigma\, a_n$ is **absolutely convergent** if $\Sigma\, |a_n|$ converges.
2. $\Sigma\, a_n$ is **conditionally convergent** if $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges.

Ex. 6: Determine the convergence or divergence of the series: $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

Let $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}} = \sum_{n=0}^{\infty} (-1)^n a_n$, with $\frac{1}{\sqrt{n+4}} = a_n$.

Let's use the Alternating Series Test to show convergence for the series, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$. We need to show $a_n > 0$. We know $1 > 0$ and $\sqrt{n+4} > 0$ for all $n \geq 1$. We can see that $\frac{1}{\sqrt{n+4}} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $a_n > 0$ for all $n \geq 1$.

Consider $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{\sqrt{n+4}}$

$$= 0.$$

Now, we need to show $a_{n+1} \leq a_n$ for all $n \geq 1$.

We know $n+4 \leq n+5$ for all $n \geq 1$.

we can see $\sqrt{n+4} \leq \sqrt{n+5}$

$$\frac{1}{(\sqrt{n+4})(\sqrt{n+5})}\left(\frac{\sqrt{n+4}}{1}\right) \leq \frac{1}{(\sqrt{n+4})(\sqrt{n+5})}\left(\frac{\sqrt{n+5}}{1}\right)$$

$$\frac{1}{\sqrt{n+5}} \leq \frac{1}{\sqrt{n+4}}$$

$$\frac{1}{\sqrt{(n+1)+4}} \leq \frac{1}{\sqrt{n+4}}$$

and $a_{n+1} \leq a_n$.

Therefore, according to the Alternating Series Test, the series, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$, converges. ∎

More Ex. 6:

Does $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+4}} \right|$ converge? If it does, then we absolute convergence, if it diverges, then we say $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$ converges conditionally.

Since $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{\sqrt{n+4}} \right| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$, we can look at the graph of the sequence of partial sums and see that there is no asymptotic behavior. Also, we can "think" $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \approx \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

This is a divergent p-series with $p = \frac{1}{2}$.

Let's use the Limit Comparison Test to show that the series, $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$, diverges. Let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, with $\frac{1}{\sqrt{n}} = b_n$, and let $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} = \sum_{n=1}^{\infty} a_n$, with $\frac{1}{\sqrt{n+3}} = a_n$.

We need to show $a_n > 0$ and $b_n > 0$. We know $1 > 0$ and $\sqrt{n+3} > 0$ for all $n \geq 1$. We can see that $\frac{1}{\sqrt{n+3}} > 0$, since a ratio of positive numbers is positive. (RoPNiP) This means $a_n > 0$ for all $n \geq 1$. We know that $\sqrt{n} > 0$ for all $n \geq 1$, and we can see that $\frac{1}{\sqrt{n}} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $b_n > 0$ for all $n \geq 1$.

Consider $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = \lim_{n \to \infty} \left(\frac{\frac{1}{\sqrt{n+3}}}{\frac{1}{\sqrt{n}}} \right)$

$$= \lim_{n \to \infty} \left(\frac{1}{\sqrt{n+3}} \right) \left(\frac{\sqrt{n}}{1} \right)$$

$$= \lim_{n \to \infty} \sqrt{\frac{n}{n+3}}$$

$$= \sqrt{\lim_{n \to \infty} \left(\frac{n}{n+3} \right)}$$

$$= \sqrt{\lim_{n \to \infty} \left[\frac{\frac{n}{1}}{\frac{n+3}{1}} \right] \cdot \left[\frac{\frac{1}{n}}{\frac{1}{n}} \right]}$$

$$= \sqrt{\lim_{n \to \infty} \frac{1}{1 + \frac{3}{n}}} = \sqrt{\frac{1}{1+0}} = \underline{1.}$$

Still More Ex. 6:

Since $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=1$ is finite and positive, and since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a divergent p-series, with $p=\frac{1}{2}$, according to the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+4}}$ is also a divergent series. This means that $\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{\sqrt{n+4}}\right|$ diverges, and we say that $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+4}}$ converges conditionally.

$\square$

Ex. 7: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

Consider $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. Since this is a convergent p-series, with $p = 3/2$, we can see that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n\sqrt{n}} \right|$ converges as well. According to theorem 9.16, Absolute Convergence, when $\sum |a_n|$ converges, then the series $\sum a_n$ also converges. This means that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$ converges, and we can say that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$ is absolutely convergent. $\square$